MATH 5061 Lecture 14 (Apr 21)
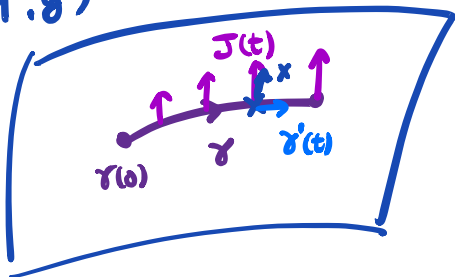
Last time:

Rauch comparison theorem

Let
- (M^n, g), $(\tilde{M}^{n+k}, \tilde{g})$ be complete Riem. manifolds
- $\gamma: [0,a] \to M$, $\tilde{\gamma}: [0,a] \to \tilde{M}$ be geodesic in M and $\tilde{M}$ s.t. $|\gamma'| = |\tilde{\gamma}'|$ (ie. same speed)
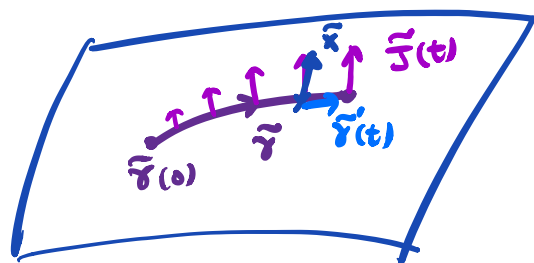- J, $\tilde{J}$ are normal Jacobi fields along γ, $\tilde{\gamma}$ respectively s.t. $J(0) = 0 = \tilde{J}(0)$, $|J'(0)| = |\tilde{J}'(0)|$

ASSUME: ① $\nexists$ conjugate pt along $\tilde{\gamma}$ on $[0,a]$

② $\forall t \in [0,a]$, $\forall x \in T_{\gamma(t)}M$, $\forall \tilde{x} \in T_{\tilde{\gamma}(t)}\tilde{M}$.

we have $K_{\gamma(t)}\left(\text{span}\{\gamma'(t), x\}\right) \leq \tilde{K}_{\tilde{\gamma}(t)}\left(\text{span}\{\tilde{\gamma}'(t), \tilde{x}\}\right)$

THEN: $|J(t)| \geq |\tilde{J}(t)|$ $\forall t \in [0,a]$

Last week, we proved the

Index Lemma: On a geodesic segment without conjugate points,

$$I(J, J) \leq I(V, V) \quad \text{where} \quad J \text{ and } V \text{ agree at the end points and } J \text{ is a Jacobi field.}$$

Proof of Rauch comparison Thm:

WLOG, assume $|J'(0)| = |\tilde{J}'(0)| > 0$.

Denote: $v(t) := |J(t)|^2$ and $\tilde{v}(t) := |\tilde{J}(t)|^2$

Goal: $v(t) \geq \tilde{v}(t) \quad \forall t \in [0, a]$

Observe, by assumption ①, $\tilde{v}(t) > 0 \quad \forall t \in (0, a]$. So we define

$$h(t) := \frac{v(t)}{\tilde{v}(t)} \qquad t \in (0, a] \qquad \text{Goal: } h(t) \geq 1 \quad \forall t \in (0, a]$$
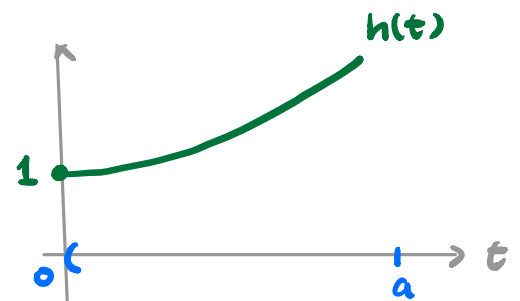
By L'Hospital's Rule,

$$v'(t) = 2\langle J(t), J'(t) \rangle$$
$$v''(t) = 2\langle J'(t), J'(t) \rangle + 2\langle J(t), J''(t) \rangle$$

$$\lim_{t \to 0} h(t) = \lim_{t \to 0} \frac{v'(t)}{\tilde{v}'(t)} = \lim_{t \to 0} \frac{v''(t)}{\tilde{v}''(t)} = \frac{|J'(0)|^2}{|\tilde{J}'(0)|^2} = 1$$

Claim: $h'(t) \geq 0 \quad \forall t \in (0, a)$

ie. $v'\tilde{v} \geq v\tilde{v}' \quad \forall t \in (0, a)$ (*)

h(t)

Pf of Claim: Fix any $t_0 \in (0, a)$.

Case 1: $v(t_0) = 0 \quad (\Rightarrow J(t_0) = 0)$

Then, $v'(t_0) = 2\langle J(t_0), J'(t_0) \rangle = 0$

Clearly, (*) holds.

Case 2: $v(t_0) \neq 0$ $\quad (\Rightarrow v(t_0) = |J(t_0)|^2 > 0)$

By rescaling, we obtain two new Jacobi fields

$$U(t) := \frac{1}{\sqrt{v(t_0)}} J(t) \quad ; \quad \tilde{U}(t) := \frac{1}{\sqrt{\tilde{v}(t_0)}} \tilde{J}(t)$$

Observe: $\quad \dfrac{v'(t_0)}{v(t_0)} = 2 I_{t_0}(U,U) \quad \underline{AND} \quad \dfrac{\tilde{v}'(t_0)}{\tilde{v}(t_0)} = 2 I_{t_0}(\tilde{U},\tilde{U})$

Why? $\quad 2 I_{t_0}(U,U) := 2 \displaystyle\int_0^{t_0} \langle U', U' \rangle - R(\gamma', U, \gamma', U) \, dt$

$$= \int_0^{t_0} (\langle U, U \rangle)'' \, dt \overset{by}{\underset{part}{=}} \langle U, U \rangle'(t_0)$$

$$= 2 \langle U(t_0), U'(t_0) \rangle = 2 \frac{\langle J(t_0), J'(t_0) \rangle}{v(t_0)} = \frac{v'(t_0)}{v(t_0)}$$

So, it suffices to show

$$\boxed{I_{t_0}(\tilde{U}, \tilde{U}) \leq I_{t_0}(U, U)} \quad (*)$$

Fix **parallel** O.N.B. along $\gamma, \tilde{\gamma}$:

$$\left\{ e_1 = \frac{\gamma'}{|\gamma'|}, e_2, \dots, e_n \right\}(t) \text{ for } T_{\gamma(t)} M^n \text{ s.t. } e_2(t_0) = U(t_0)$$

$$\left\{ \tilde{e}_1 = \frac{\tilde{\gamma}'}{|\tilde{\gamma}'|}, \tilde{e}_2, \dots, \tilde{e}_{n+k} \right\}(t) \text{ for } T_{\tilde{\gamma}(t)} \tilde{M}^{n+k} \text{ s.t. } \tilde{e}_2(t_0) = \tilde{U}(t_0)$$

Define a map $\phi : \left\{ \begin{array}{c} \text{vector field} \\ \text{along } \gamma(t) \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{vector field} \\ \text{along } \tilde{\gamma}(t) \end{array} \right\}$

$$V(t) = \sum_{i=1}^{n} \boxed{g_i(t)} e_i(t) \overset{\phi}{\longmapsto} (\phi V)(t) := \sum_{i=1}^{n} \boxed{g_i(t)} \tilde{e}_i(t)$$

same coefficients

Properties of ϕ :

- $\langle \phi V, \phi V \rangle = \langle V, V \rangle \qquad \forall t \in [0,a]$

- $(\phi V)' = \phi(V') \qquad \forall t \in [0,a]$

*. $I_{t_0}(\phi V, \phi V) \leq I_{t_0}(V, V)$

Reason:
$$I_{t_0}(V,V) := \int_0^{t_0} |V'(t)|^2 - R(\gamma', V, \gamma', V)_{(t)} \, dt$$

$$\wedge \text{\| by first 2 properties} \qquad \wedge \text{ by 1st property and } K \leq \tilde{K}$$

$$I_{t_0}(\phi V, \phi V) := \int_0^{t_0} |(\phi V)'|^2 - \tilde{R}(\tilde{\gamma}', \phi V, \tilde{\gamma}', \phi V) \, dt$$

Recall by construction, we have

$$(\phi \tilde{U})(t_0) = \phi(e_2(t_0)) = \tilde{e}_2(t_0) = \tilde{U}(t_0)$$

i.e. $\phi \tilde{U}$ and $\tilde{U}$ have the same boundary values along γ on $[0, t_0]$

By (*) and Index Lemma,

$$\frac{1}{2} \frac{\tilde{v}'(t_0)}{\tilde{v}(t_0)} = I_{t_0}(\tilde{U}, \tilde{U}) \leq I_{t_0}(\phi \tilde{U}, \phi \tilde{U}) \overset{(\ast)}{\leq} I_{t_0}(U, U) = \frac{1}{2} \frac{v'(t_0)}{v(t_0)}$$

Volume comparison via Jacobi field estimates

Thm: (Bishop-Gromov & Gunther)

Let (M^n, g) be a complete Riem. mfd.

Fix $p \in M$, let $a, b \in \mathbb{R}$ be constants.

(i) If $\mathrm{Ric}_x^M(u,u) \geq (n-1) a \quad \forall x \in M, \forall u \in T_xM, |u| = 1,$

$$\text{then} \quad \mathrm{Vol}(B_p^M(r)) \leq \mathrm{Vol}(B^{\mathcal{M}_a}(r)) \quad \forall r \in (0, \mathrm{inj}_p(M)).$$

Here, $B^{\mathcal{M}_a}(r) = $ Volume of ball of radius r

inside a space form $\mathcal{M}_a$ of const. curvature a

(ii) If $K^M \leq b$, then

$$\mathrm{Vol}(B_p^M(r)) \geq \mathrm{Vol}(B^{\mathcal{M}_b}(r)) \quad \forall r \in (0, \mathrm{inj}_p(M)).$$

Digression: Jacobi field and Volume estimates

Volume form of (M^n, g) :
$$dV_g = \sqrt{\det(g_{ij})} \, dx^1 \wedge \cdots \wedge dx^n$$
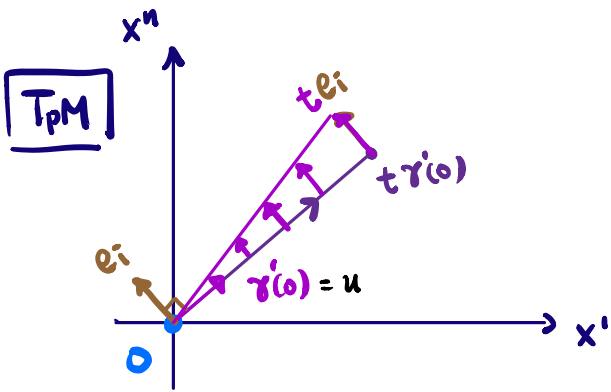in local coord. $(x^1, \ldots, x^n)$.

and
$$\mathrm{Vol}(\mathcal{U}) = \int_{\mathcal{U}}^{M} \sqrt{\det(g_{ij})} \, dx^1 \cdots dx^n$$

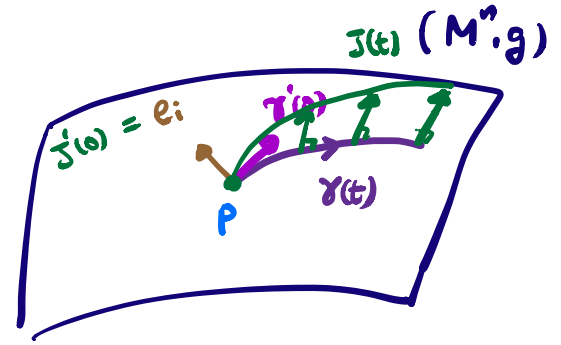In particular, in geodesic normal coordinates (based at $p \in M$)

$$\mathrm{Vol}(B_p^M(r)) = \int_{S^{n-1}} \int_0^r J(u,t) \cdot t^{n-1} \, dt \, du$$

where $J(u,t) := t^{-(n-1)} \sqrt{\det \langle J_i(t), J_j(t) \rangle}$

some Jacobi fields

Picture:

Fix $u \in T_p M$, $|u| = 1$, and a geodesic $\gamma(t) = \exp_p(tu)$

Fix an O.N.B. $\{u = e_1, e_2, ..., e_n\}$ of $T_p M$

Let $J_i(t)$ be the Jacobi field along $\gamma(t)$ s.t

$$J_i(0) = 0 \quad \text{and} \quad J_i'(0) = e_i$$

i.e $\quad J_i(t) = d(\exp_p)_{tu}(t e_i)$

So, $\quad dV_g = \sqrt{\det(g_{ij})} \, dx^1 \cdots dx^n$

$$= \sqrt{\det\left(\underbrace{d\exp(e_i)}_{\frac{1}{t} J_i(t)}, \underbrace{d\exp(e_j)}_{\frac{1}{t} J_j(t)}\right)} \, t^{n-1} \, dt \, du$$

$$= J(u,t) \, t^{n-1} \, dt \, du$$

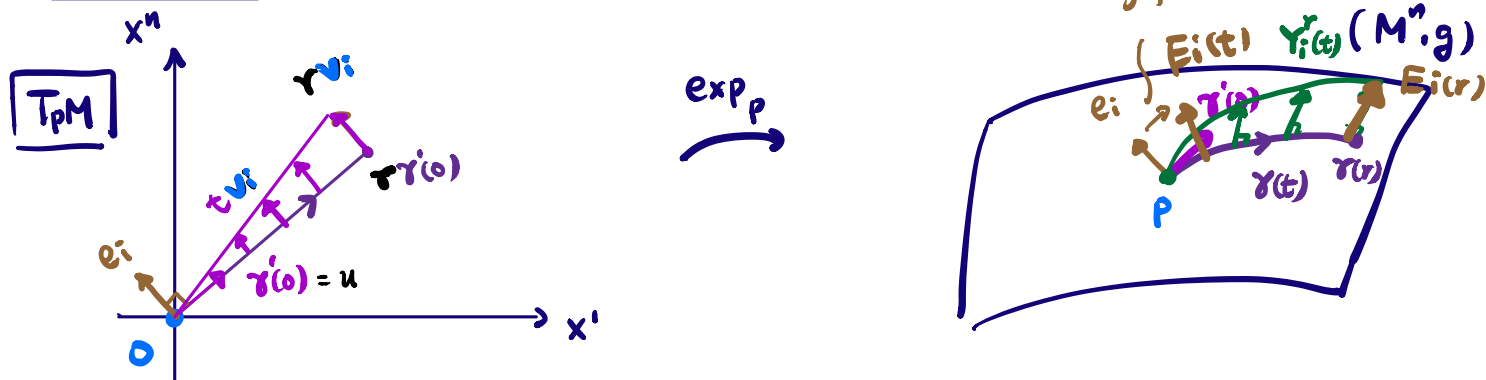On the other hand, we have explicit formula for the Jacobi fields in space forms M_a

Jacobi eq^n : $\quad J'' + a J = 0 \quad \Rightarrow \quad J(t) = S_a(t) \, E(t)$
$\qquad\qquad\qquad$ and $J(0) = 0 \qquad\qquad\qquad\qquad$ parallel v.f.

where $\quad S_a(t) := \begin{cases} \sin\sqrt{a}\, t & \text{. if } a > 0 \\ t & \text{. if } a = 0 \\ \sinh\sqrt{-a}\, t & \text{. if } a < 0 \end{cases}$

Proof of Volume Comparison Thm (i) : Ric $\geq (n-1)$ a

Picture:

Denote $Y_i^r(t)$ to be the unique Jacobi field s.t.

$$Y_i^r(0) = 0 \quad \text{and} \quad Y_i^r(r) = E_i(r)$$

and $V_i \in T_pM$ s.t. $d(exp_p)_{tu}(t V_i) = Y_i^r(t)$

Note: $J(u,t) = C_r \, t^{1-n} \, \det(Y_2^r(t), \ldots, Y_n^r(t))$

(Ex)

where $C_r = \left[\det(Y_2'^r(0), \ldots, Y_n'^r(0)) \right]$

Denote: For each $u \in T_pM$, $|u| = 1$,

$$f(t) := J(u,t)$$

Lemma: $\dfrac{f'(r)}{f(r)} = \sum_{i=2}^{n} I_r(Y_i^r, Y_i^r) - \dfrac{n-1}{r}$

Proof of Lemma:

Let $g(t) := \det(\langle Y_i^r(t), Y_j^r(t) \rangle)$. Then

$$f(t) := J(u,t) := C_r \, t^{1-n} \, \det(Y_2^r(t), \ldots, Y_n^r(t)) = C_r \, t^{1-n} \sqrt{g(t)}$$

$$\Rightarrow \quad f'(t) = C_r \, t^{1-n} \cdot \frac{1}{2} \frac{g'(t)}{\sqrt{g(t)}} - C_r (n-1) t^{-n} \sqrt{g(t)}$$

$$\Rightarrow \quad \frac{f'(t)}{f(t)} = \frac{g'(t)}{2g(t)} - \frac{n-1}{t}$$

On the other hand, we want $\dfrac{g'(r)}{2g(r)} = \displaystyle\sum_{i=2}^{n} I_r(Y_i^r, Y_i^r)$.

Note: $g(r) := \det\left(\langle Y_i^r(r), Y_j^r(r) \rangle \right) = \det \langle E_i(r), E_j(r) \rangle = 1$

and $g'(r) = 2 \displaystyle\sum_{i=2}^{n} \langle (Y_i^r)'(r), Y_i^r(r) \rangle$

By the proof of Claim of Index Lemma, $I_r(Y_i^r, Y_i^r) = \langle Y_i^r(r), Y_i^{r\,\prime}(r) \rangle$.

$$\underline{\text{Lemma}} \qquad \square$$

To finish the proof of **Volume Comparison Thm (i)**,

Consider the vector field along γ

$$X_i^r(t) := \frac{S_a(t)}{S_a(r)} E_i(t)$$

s.t. $X_i^r(0) = 0 = Y_i^r(0)$ and $X_i^r(r) = E_i(r) = Y_i^r(r)$.

By **Index Lemma**, $I_r(Y_i^r, Y_i^r) \leq I_r(X_i^r, X_i^r)$.

Note: $I_r(X_i^r, X_i^r) = \displaystyle\int_0^r \left(\frac{S_a'(t)}{S_a(r)} \right)^2 - \left(\frac{S_a(t)}{S_a(r)} \right)^2 R(\gamma', E_i, \gamma', E_i) \, dt$

$$= \int_0^r \left(\frac{S_a(t)}{S_a(r)} \right)^2 \underbrace{[a - R(\gamma', E_i, \gamma', E_i)]}_{\sum_{i=1}^n (\ ) \leq 0 \text{ by assumption}} dt + \frac{S_a'(r)}{S_a(r)}$$

$$\Rightarrow \quad \frac{f'(r)}{f(r)} = \sum_{i=2}^{n} I_r(Y_i^r, Y_i^r) - \frac{n-1}{r} \leq \sum_{i=2}^{n} I_r(X_i^r, X_i^r) - \frac{n-1}{r}$$

$$\leq (n-1)\left(\frac{S_a'(r)}{S_a(r)} - \frac{1}{r} \right)$$

Integrate in r $\Rightarrow$ $f(r) \leq S_a(r)$. $\Rightarrow$ **DONE!**

integrate again

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$$